TUTORIAL NOTES FOR MATH4220

JUNHAO ZHANG

1. GREEN'S FUNCTIONS AND ELLIPTIC EQUATIONS

Let us recall some useful facts for solving the problems of elliptic equations.

Theorem 1 (Green's identity)**.** *Suppose $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Then for any $x \in \Omega$, there holds*

$$u(x) = \int_{\Omega} \Gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega} \left(\Gamma(x,y)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial \Gamma}{\partial n}(x,y) \right) dS_y,$$

where

$$\Gamma(x,y) = \begin{cases} \frac{1}{2\pi}\log|x-y|, & n=2, \\ -\frac{1}{4\pi}|x-y|^{-1}, & n=3. \end{cases}$$

For the Dirichlet boundary value problem,

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in \Omega, \\ u(x) &= \varphi(x), \quad x \in \partial\Omega, \end{aligned}$$

It suffices to find the Green's function $G(x,y) = \Gamma(x,y) + \Psi(x,y)$ where

$$\begin{aligned} \Delta_y \Psi(x,y) &= 0, \quad y \in \Omega, \\ \Psi(x,y) &= -\Gamma(x,y), \quad y \in \partial\Omega, \end{aligned}$$

then

$$u(x) = \int_{\Omega} G(x,y)f(y)dy + \int_{\partial\Omega} \varphi(y)\frac{\partial G}{\partial n}(x,y)dS_y.$$

In the following, we discuss some examples concerning the Green's function.

Problem 2. Find the Green's function for Dirichlet boundary value problem over the first quadrant

$$\mathbb{R}^2_+ = \{x = (x_1, x_2) : x_1 > 0, x_2 > 0\}.$$

Solution. The Green's function is

$$\begin{aligned} G(x,y) =& \frac{1}{2\pi}\log\sqrt{|x_1-y_1|^2+|x_2-y_2|^2} - \frac{1}{2\pi}\log\sqrt{|x_1+y_1|^2+|x_2-y_2|^2} \\ &+ \frac{1}{2\pi}\log\sqrt{|x_1+y_1|^2+|x_2+y_2|^2} - \frac{1}{2\pi}\log\sqrt{|x_1-y_1|^2+|x_2+y_2|^2}. \end{aligned}$$

Problem 3. Find the Green's function for Dirichlet boundary value problem over the half disk

$$B_R^+ = \{x = (x_1, x_2) : |x_1|^2 + |x_2|^2 \le R^2, x_2 > 0\}.$$

Solution. The Green's function is

$$G(x,y) =\frac{1}{2\pi}\left(\log\sqrt{|x_1-y_1|^2+|x_2-y_2|^2}-\log\sqrt{|\frac{R}{|x|}x_1-\frac{|x|}{R}y_1|^2+|\frac{R}{|x|}x_2-\frac{|x|}{R}y_2|^2}\right)$$
$$-\frac{1}{2\pi}\left(\log\sqrt{|x_1-y_1|^2+|x_2+y_2|^2}-\log\sqrt{|\frac{R}{|x|}x_1-\frac{|x|}{R}y_1|^2+|\frac{R}{|x|}x_2+\frac{|x|}{R}y_2|^2}\right).$$

A Supplementary Problem

Problem 4. Show that for the Neumann boundary value problem,

$$-\Delta u(x) = f(x), \quad x\in\Omega,$$
$$\frac{\partial u}{\partial n}(x) = \varphi(x), \quad x\in\partial\Omega,$$

If there exists a Green's function $G(x,y) = \Gamma(x,y) + \Psi(x,y)$ where

$$\Delta_y\Psi(x,y) = \frac{1}{|\Omega|}, \quad y\in\Omega,$$
$$\frac{\partial\Psi}{\partial n}(x,y) = -\frac{\partial\Gamma}{\partial n}(x,y), \quad y\in\partial\Omega,$$

then

$$u(x)-\frac{1}{|\Omega|}\int_\Omega u(y)dy = \int_\Omega G(x,y)f(y)dy + \int_{\partial\Omega}\varphi(y)\frac{\partial G}{\partial n}(x,y)dS_y.$$

Moreover, if $\Omega = \mathbb{R}^n_+ := \{x = (x_1, ..., x_n) : x_n > 0\}$, then

$$G(x,y) = \begin{cases} -\frac{1}{2\pi}(\log|x-y| - \log|x-y|), & n=2, \\ -\frac{1}{4\pi}(|x-y|^{-1} + |x-y|^{-1}), & n=3. \end{cases}$$

If $\Omega = B_R := \{x = (x_1, ..., x_n) : \sum_{i=1}^n |x_n|^2 \le R^2\}$, then

$$G(x,y) = \begin{cases} -\frac{1}{2\pi}\left(\log\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right| + \log|x-y|\right) - \frac{1}{4\pi R^2}|y|^2, & n=2, \\ \frac{1}{4\pi}\left(\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|^{-1} + |x-y|^{-1}\right) + \frac{1}{4\pi}\log\left[\left(\frac{R}{|x|}x-\frac{|x|}{R}y\right)\cdot\frac{Rx}{|x|^2} + \left|\frac{R^2}{|x|^2}x-y\right|\right] - \frac{1}{8\pi R^3}|y|^2, & n=3. \end{cases}$$

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

Email address: jhzhang@math.cuhk.edu.hk